MODULI SPACES OF CURVES WITH LINEAR SERIES AND THE SLOPE CONJECTURE

DEEPAK KHOSLA

ABSTRACT. We describe the moduli space $\mathcal{G}_d^r$ of triples consisting of a curve C, a line bundle L on C of degree d, and a linear system V on L of dimension r. This moduli space extends over a partial compactification $\widetilde{\mathcal{M}}_g$ of $\mathcal{M}_g$ inside $\overline{\mathcal{M}}_g$. For the proper map $\eta\colon \mathcal{G}_d^r \to \widetilde{\mathcal{M}}_g$, we compute the push-forward on Chow 1-cocyles in the case where η has relative dimension zero. As a consequence we obtain another counterexample to the Harris-Morrison slope conjecture as well as an infinite sequence of potential counterexamples.

1. Introduction

One of the fundamental objects in the study of the birational geometry of a projective variety is its cone of effective divisors. Deligne and Mumford [2] constructed a compactification $\overline{\mathcal{M}}_g$ of the moduli space $\mathcal{M}_g$ of smooth genus-g algebraic curves. By a theorem of Harer [10] in topology, for $g \geq 3$,

$$\operatorname{Pic} \overline{\mathcal{M}}_g \otimes \mathbf{Q} = \mathbf{Q}\lambda \oplus \mathbf{Q}\delta_0 \oplus \mathbf{Q}\delta_1 \oplus \cdots \oplus \mathbf{Q}\delta_{\lfloor g/2 \rfloor},$$

where λ comes from a class on $\mathcal{M}_g$ and the δ_i are boundary classes. It is natural to ask the question: which linear combinations of λ and the δ_i are effective? If we consider only classes of the form $a\lambda - b\delta$, where

$$\delta = \delta_0 + \delta_1 + \cdots + \delta_{\lfloor g/2 \rfloor}$$

is the total boundary class, then the shape of the effective cone on this plane is determined by the single number

$$s_g = \inf\left\{ \frac{a}{b} \mid a, b \geq 0,\ a\lambda - b\delta \text{ is effective} \right\},$$

which is known as the *slope* of $\overline{\mathcal{M}}_g$. By explicitly constructing effective divisors with small slope, Harris and Mumford [13] deduced that $\mathcal{M}_g$ is of general type when $g \geq 24$. Conversely, a suitable lower bound on the slope of $\overline{\mathcal{M}}_g$ would force the Kodaira dimension of $\mathcal{M}_g$ to be $-\infty$. In 1990 Harris and Morrison [12] conjectured that

$$s_g \geq 6 + \frac{12}{g+1}.$$

Recently Farkas and Popa [6] found a counterexample to this conjecture in genus 10. Specifically they considered the divisor

$$D = \{[C] \in \mathcal{M}_{10} \mid C \text{ lies on a K3 surface}\}$$

and showed that its class is

$$7\lambda - \delta_0 - 5\delta_1 - \cdots.$$

This has a slope of 7, which is less than $6 + \frac{12}{11}$. Farkas has gone further and shown that $\mathcal{M}_g$ is of general type when g is 22 or 23 [5]. Though the divisor D appears to be an isolated example, in fact, it may be re-written as the divisorial component of

> the locus of $[C] \in \mathcal{M}_{10}$ for which there is an embedding $C \hookrightarrow \mathbf{P}^4$ of degree 12 such that C lies on a quadric.

This suggests a host of generalizations. For example, we may define $D_{d,g}^{r,k}$ to be

> the locus of $[C] \in \mathcal{M}_g$ for which there is an embedding $C \hookrightarrow \mathbf{P}^r$ of degree d such that C lies on a hypersurface of degree k.

The problem is then to determine when such cycles are divisors and, in those cases, to compute their classes. Our approach is to consider the moduli space $\mathcal{G}_d^r(\mathcal{M}_g)$ of *curves with linear series*. We set

$$\mathcal{G}_d^r(\mathcal{M}_g) = \mathcal{G}_d^r = \left\{ (C, L, V) \mid [C] \in \mathcal{M}_g, L \in \operatorname{Pic}^d C, V \subset H^0(L) \right\},$$

where V is a subspace of dimension $r + 1$. Then $D_{d,g}^{r,k}$ is naturally the image of the subscheme $\widetilde{D}$ in $\mathcal{G}_d^r$,

$$\widetilde{D} = \left\{ (C, L, V) \mid \operatorname{Sym}^k V \to H^0(L^{\otimes k}) \text{ has a kernel} \right\}$$

If it is of the expected dimension, the class of $\widetilde{D}$ is then easily evaluated in terms of certain standard classes on $\mathcal{G}_d^r$. It remains to compute the proper push-forward map for the morphism

$$\eta \colon \mathcal{G}_d^r \to \mathcal{M}_g.$$

In our main result, Theorem 3.5, we compute η_* on divisors in the case where η has relative dimension 0.

In Section 2 we work out the case where $g = 21$, $r = 6$, $d = 24$, and $k = 2$, so that the cover

$$\mathcal{G}_{24}^6 \to \mathcal{M}_{21}$$

is generically finite, and the condition of lying on a quadric imposes one condition on a $\mathfrak{g}_{24}^6$. We show that $D_{24,21}^{6,2}$ has a divisorial component and use Theorem 3.5 to compute its class. It turns out that $D_{24,21}^{6,2}$ provides another counterexample to the Slope Conjecture. In fact, one may consider

$$(g, r, d) = \big(m(2m+1), 2m, 2m(m+1)\big)$$

for any integer $m \geq 1$. In this case the Brill-Noether number ρ is zero, and it is one condition to lie on a quadric. Although at this time we do not know how to prove that the $D_{d,g}^{r,k}$ are divisors for all m, our calculations show that if they were, they would all provide counter-examples to the Slope Conjecture.

In Section 3 we outline the definition and basic properties of the moduli space $\mathcal{G}_d^r$ before stating the main theorem. In Section 4 we give the statements of a series of calculations over special families of stable curves. These calculations assemble to give the main result. Finally Section 5 is devoted to the proofs of the lemmas stated in Section 4.

Notation 1. Unless otherwise indicated, all schemes are of finite type over $\mathbf{C}$. For a scheme (or Deligne-Mumford stack) X of dimension n, we write $A^k(X)$ for the Chow group $A_{n-k}(X)$.

2. A Generalized Brill-Noether Divisor

In this section we work on $\mathcal{M}_{21}^{\mathrm{irr}}$, the locus of irreducible stable genus-21 curves, and we let $\mathcal{G}_{24}^6$ denote $\mathcal{G}_{24}^6(\mathcal{M}_{21}^{\mathrm{irr}})$. This space may be informally defined as the locus of triples

$$\left\{ (C, L, V) \mid [C] \in \mathcal{M}_{21}^{\mathrm{irr}}, L \in \overline{\mathrm{Pic}}^{24} C, V \subset H^0(L) \right\},$$

where L is allowed to be a torsion-free sheaf of rank 1. In Section 3 we will give a precise definition of this space, which differs slightly from the above.

We first establish the following result.

Lemma 2.1. *The space $\mathcal{G}_{24}^6(\mathcal{M}_{21})$ is irreducible.*

Proof. Note that a $\mathfrak{g}_{24}^6$ is residual to a $\mathfrak{g}_{16}^2$; that is

$$L \in W_{24}^6(C) \iff K_C \otimes L^* \in W_{16}^2(C)$$

for any smooth curve C of genus 21. Thus there is a dominant rational map

$$V_{16,21} \longrightarrow \mathcal{G}_{24}^6$$

from the Severi variety of irreducible plane curves of degree 16 and genus 21. Since $V_{16,21}$ is irreducible [11] and maps dominantly to $\mathcal{G}_{24}^6$, so $\mathcal{G}_{24}^6$ is irreducible. $\qquad\square$

If $\pi \colon \mathcal{C}_{24}^6 \to \mathcal{G}_{24}^6$ is the universal curve, we let $\mathcal{L} \to \mathcal{C}_{24}^6$ be a universal line bundle and $\mathcal{V} \subset \pi_* \mathcal{L}$ the universal rank-7 subbundle.

Definition 2.2. Consider the open set $U \subset \mathcal{G}_{24}^6$ over which $\mathcal{L}$ is a line bundle. Over U there is a map m of vector bundles of rank 28,

$$m \colon \mathrm{Sym}^2 \mathcal{V} \to \pi_* \mathcal{L}^{\otimes 2}$$

Let $\widetilde{E}$ be the closure in $\mathcal{G}_{24}^6$ of the singular locus of m. Since the complement of U has codimension 2, the class of the degeneracy locus in U extends uniquely to $\mathcal{G}_{24}^6$.

Proposition 2.3. *The scheme $\widetilde{E}$ has codimension 1 inside the irreducible component of $\mathcal{G}_{24}^6$ which dominates $\mathcal{M}_{21}$.*

Proof. Since $\mathcal{G}_{24}^6(\mathcal{M}_{21})$ is irreducible, it suffices to exhibit a smooth curve with a $\mathfrak{g}_{24}^6$ not lying on a quadric. Let

$$S = \mathrm{Bl}_{21} \mathbf{P}^2$$

be the blow-up of $\mathbf{P}^2$ at 21 general points, and consider the linear system

$$\nu = \left| 13H - 2 \sum_{j=1}^{9} E_j - 3 \sum_{k=10}^{21} E_k \right|$$

on S, where H is the hyperplane class and E_i are the exceptional divisors. A calculation in Macaulay (see the appendix) shows that a general member C of ν is irreducible and smooth of genus 21. The series

$$\left| 6H - \sum_{i=1}^{21} E_i \right|$$

embeds S in $\mathbf{P}^6$ as the rank-2 locus of general 3×6 matrix of linear forms [3, Section 20.4]. The ideal of S is therefore generated by cubics, so S does not lie on quadric. It follows that C, which embeds in $\mathbf{P}^6$ in degree 24, does not lie on a quadric. $\qquad\square$

Definition 2.4. Let E be the effective codimension-1 Chow cycle which is the image of $\widetilde{E}$ under the map

$$\eta \colon \mathcal{G}_{24}^6 \to \mathcal{M}_{21}^{\mathrm{irr}}$$

To compute the class of E, we begin by expressing the class of $\widetilde{E}$ in terms of simpler classes on $\mathcal{G}_{24}^6$. Let

$$\alpha = \pi_* c_1(\mathcal{L})^2$$
$$\beta = \pi_* c_1(\mathcal{L}) \cdot c_1(\omega)$$
$$\gamma = c_1(\mathcal{V})$$

Proposition 2.5. *The class of $\widetilde{E} \subset \mathcal{G}_{24}^6$ is*

$$2\alpha - \beta + \lambda - 8\gamma$$

Proof. By Porteous, the class of $\widetilde{E}$ is

$$c_1(\pi_* \mathcal{L}^{\otimes 2}) - c_1(\mathrm{Sym}^2 \mathcal{V})$$

By Grothendieck-Riemann-Roch applied to the projection $\pi \colon \mathcal{C}_{24}^6 \to \mathcal{G}_{24}^6$ from the universal curve,

$$
\begin{aligned}
\mathrm{ch}(\pi_* \mathcal{L}^{\otimes 2}) &= \pi_* \left[\mathrm{ch}(\mathcal{L}^{\otimes 2}) \cdot \mathrm{td}_{\mathcal{C}_{24}^6/\mathcal{G}_{24}^6} \right] \\
&= \pi_* \left[\left(1 + 2c_1(\mathcal{L}) + 2c_1(\mathcal{L})^2 + \cdots \right) \right. \\
&\qquad\qquad \left. \cdot \left(1 - \frac{c_1(\omega)}{2} + \frac{c_1(\omega)^2 + \kappa}{12} + \cdots \right) \right] \\
&= 28 + (2\alpha - \beta + \lambda) + \cdots
\end{aligned}
$$

where ω is the relative dualizing sheaf for π, and κ is the divisor of nodes on $\mathcal{C}_{24}^6$. Also,

$$c_1(\mathrm{Sym}^2 \mathcal{V}) = 8c_1(\mathcal{V}) = 8\gamma,$$

and the proposition follows. $\qquad\square$

Proposition 2.6. *The class of $E \subset \mathcal{M}_{21}^{\mathrm{irr}}$ is given as*

$$[E] = 2459\lambda - 377\delta_0.$$

Proof. By Proposition 2.5 and Theorem 3.5,

$$
\begin{aligned}
[E] = \eta_*[\widetilde{E}] &= \eta_*(2\alpha - \beta + \lambda - 8\gamma) \\
&= \frac{2459N}{95}\lambda - \frac{377N}{95}\delta_0,
\end{aligned}
$$

where N is the degree of η. $\qquad\square$

Corollary 2.7. *The slope conjecture is false in genus 21.*

Proof. Since

$$\frac{2459}{377} < 6 + \frac{12}{22},$$

this is an immediate consequence of [6, Corollary 1.2]. $\qquad\square$

Remark 2.8. For any integer $m \geq 1$, if we let

$$(g, r, d) = \big(m(2m+1), 2m, 2m(m+1)\big),$$

then, as mentioned in the introduction, we can consider an analogous locus

$$\widetilde{E} \subset \mathcal{G}_d^r(\mathcal{M}_g^{\mathrm{irr}}).$$

At this time, we do not have a result similar to Lemma 2.1 which allows us to show that $\widetilde{E}$ is actually a divisor. Were this the case, however, identical computations to those above would show that the ratio a/b_0 of the coefficients of λ and δ_0 in E is less than $6 + 12/(g+1)$. Specifically, we compute the difference

$$6 - \frac{12}{g+1} - \frac{a}{b_0} = \frac{36m^5 - 24m^4 - 57m^3 + 48m^2 + 3m - 6}{16m^9 - 8m^8 - 4m^7 - 10m^6 + 23m^4 + 16m^3 + 13m^2 + 2m}.$$

3. Moduli of Curves with Linear Series

In this section we give the basic definitions needed to state the main theorem. We begin by constructing a partial compactification of $\mathcal{M}_g$ inside $\overline{\mathcal{M}}_g$ over which we can extend the space $\mathcal{G}_d^r(\mathcal{M}_g)$ of curves with linear series.

Definition 3.1. For $i \in \{0, 1, \ldots, g\}$, let $B_i \subset \overline{\mathcal{M}}_{g,n}$ be the locus of stable pointed curves which are the union of a smooth curve of genus i and a smooth curve of genus $g-i-1$ meeting nodally at two points. Define $\widetilde{\mathcal{M}}_{g,n}$ to be the open substack of $\overline{\mathcal{M}}_{g,n}$ which is the complement of the closure of $\bigcup B_i$.

Definition 3.2. We define the Deligne-Mumford stack $\mathcal{G}_d^r(\widetilde{\mathcal{M}}_{g,n})$ of curves with linear series as follows. Fiberwise, over each irreducible curve $[C] \in \widetilde{\mathcal{M}}_{g,n}$, we consider torsion-free rank-1 coherent sheaves L on C together with a vector subspace of the space of global sections of L. Over reducible fibers, we consider limit linear series as defined by Eisenbud-Harris in [4]. These constructions have been made functorial by Altman-Kleiman [1] and Osserman [15]. Osserman treats the case where there are at most two irreducible components and there is no monodromy among the components. The details of the general contruction will appear in a forthcoming paper [14]. An important fact is that given a reducible family of curves, the scheme structure on the open locus in $\mathcal{G}_d^r$ of refined series coincides with the natural subscheme structure inside the product of Grassmannians.

The resulting Deligne-Mumford stack $\mathcal{G}_d^{r\prime}$ may not be irreducible, even over $\mathcal{M}_{g,n}$; we let $\mathcal{G}_d^r = \mathcal{G}_d^r(\widetilde{\mathcal{M}}_{g,n})$ be the unique irreducible component of $\mathcal{G}_d^{r\prime}$ which dominates $\mathcal{M}_{g,n}$.

The morphism

$$\eta \colon \mathcal{G}_d^r \to \widetilde{\mathcal{M}}_{g,n}$$

is representable and proper. In the case where

$$\rho = g - (r+1)(g-d+r)$$

is non-negative, η is generically smooth of relative dimension ρ. Given n ramification conditions $\beta_1, \ldots, \beta_n$, the substack

$$Z_{\beta_1, \ldots, \beta_n} \subset \mathcal{G}_d^r$$

of linear series with specified ramification at the marked points has the expected generic relative dimension

$$\rho - \sum_i |\beta_i|$$

over $\widetilde{\mathcal{M}}_{g,n}$. We say that a stable marked curve $(C, p_1, \ldots, p_n)$ is *Brill-Noether general* if, for all possible sets of ramification conditions $\beta_1, \ldots, \beta_n$, the marked curve $(C, p_1, \ldots, p_n)$ lies in the dense open over which $\eta|_{Z_{\beta_1,\ldots,\beta_n}}$ is flat.

In the following we work over $\widetilde{\mathcal{M}}_{g,1}$ in order to be able to consistently define the universal line and vector bundles. If $\pi \colon \mathcal{C}_d^r \to \mathcal{G}_d^r$ is the universal curve, and $\sigma \colon \mathcal{G}_d^r \to \mathcal{C}_d^r$ is the marked section, then there is a universal coherent sheaf $\mathcal{L}$ on $\mathcal{C}_d^r$, flat over $\mathcal{G}_d^r$, with the properties that

- $\mathcal{L}$ has torsion-free rank-1 fibers
- $\mathcal{L}$ has degree d on the component of each fiber which contains the marked point and has degree 0 on all other components
- $\mathcal{L}$ is trivialized along the marked section: $\sigma^*\mathcal{L} \simeq \mathcal{O}_{\mathcal{G}_d^r}$
- $\mathcal{L}$ is locally free outside a locus of codimension 3.

There is a sub-bundle

$$\mathcal{V} \hookrightarrow \pi_*\mathcal{L}$$

which, over each point in $\mathcal{G}_d^r$, is equal to the aspect of the $\mathfrak{g}_d^r$ on the component containing the marked point.

Remark 3.3. By a theorem of Harer [10], for $g \geq 3$,

$$\operatorname{Pic} \widetilde{\mathcal{M}}_{g,1} \otimes \mathbf{Q} = \mathbf{Q}\lambda \oplus \mathbf{Q}\delta_0 \oplus \mathbf{Q}\delta_1 \oplus \cdots \oplus \mathbf{Q}\delta_{g-1} \oplus \mathbf{Q}\psi$$

where λ and ψ are the first Chern classes of the Hodge and tautological bundles respectively, δ_0 is the divisor of irreducible nodal curves, and δ_i is the divisor of unions of curves of genus i and $g-i$, where the marked point lies on the component of genus i.

Definition 3.4. Since $\mathcal{L}$ is locally free in codimension 3, the cycle classes

$$c_1(\mathcal{L})^2 \cap [\mathcal{C}_d^r] \quad \text{and} \quad c_1(\mathcal{L}) \cdot c_1(\omega) \cap [\mathcal{C}_d^r]$$

on $\mathcal{C}_d^r$ are well-defined, where $\omega = \omega_{\mathcal{C}_d^r/\mathcal{G}_d^r}$ is the relative dualizing sheaf. We define cycle classes $\alpha, \beta, \gamma \in A^1(\mathcal{G}_d^r)$ as follows.

$$\alpha = \pi_*\big(c_1(\mathcal{L})^2 \cap [\mathcal{C}_d^r]\big)$$
$$\beta = \pi_*\big(c_1(\mathcal{L}) \cdot c_1(\omega) \cap [\mathcal{C}_d^r]\big)$$
$$\gamma = c_1(\mathcal{V}) \cap [\mathcal{G}_d^r].$$

We now state our main result.

Theorem 3.5. *Let $g \geq 1$, $r \geq 0$, and $d \geq 1$ be integers for which*

$$\rho = g - (r+1)(g-d+r) = 0$$

and consider the map

$$\eta \colon \mathcal{G}_d^r(\widetilde{\mathcal{M}}_{g,1}) \to \widetilde{\mathcal{M}}_{g,1}$$

If

$$\eta_* \colon A^1(\mathcal{G}_d^r(\widetilde{\mathcal{M}}_{g,1})) \to A^1(\widetilde{\mathcal{M}}_{g,1})$$

is the proper push-forward on the corresponding Chow groups, then

$$\frac{6(g-1)(g-2)}{dN}\eta_*\alpha = 6(gd - 2g^2 + 8d - 8g + 4)\lambda$$
$$+ (2g^2 - gd + 3g - 4d - 2)\delta_0$$
$$+ 6\sum_{i=1}^{g-1}(g-i)(gd + 2ig - 2id - 2d)\delta_i$$
$$- 6d(g-2)\psi,$$

$$\frac{2(g-1)}{dN}\eta_*\beta = 12\lambda - \delta_0 + 4\sum_{i=1}^{g-1}(g-i)(g-i-1)\delta_i - 2(g-1)\psi,$$

$$\frac{2(g-1)(g-2)}{N}\eta_*\gamma = \big[-(g+3)\xi + 5r(r+2)\big]\lambda - d(r+1)(g-2)\psi$$
$$+ \frac{1}{6}\big[(g+1)\xi - 3r(r+2)\big]\delta_0$$
$$+ \sum_{i=1}^{g-1}(g-i)\big[i\xi + (g-i-2)r(r+2)\big]\delta_i,$$

where

$$N = \frac{1! \cdot 2! \cdot 3! \cdots r! \cdot g!}{(g-d+r)!(g-d+r+1)! \cdots (g-d+2r)!}$$

and

$$\xi = 3(g-1) + \frac{(r-1)(g+r+1)(3g-2d+r-3)}{g-d+2r+1}.$$

The proof of Theorem 3.5 will occupy the remaining two sections.

4. Special Families of Curves

Our strategy for proving Theorem 3.5 will be to pull back to various families of stable curves over which the space of linear series is easier to analyze. In Section 4-A we define three families of pointed curves, and in Section 4-B we compute η_* for these special families. In Section 4-C we compute the pull-backs of the standard divisor classes on $\widetilde{\mathcal{M}}_{g,1}$ to the base spaces of each of our families. Assembling the results of these three sections, we compute η_* over the whole moduli space.
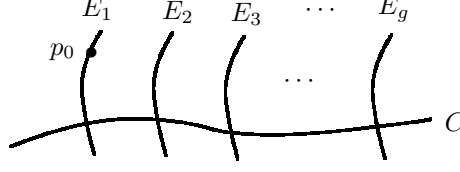
4-A. Definitions of Families.

Definition 4.1. Let $i\colon \overline{\mathcal{M}}_{0,g} \hookrightarrow \widetilde{\mathcal{M}}_{g,1}$ be the family of marked stable curves defined by sending a g-pointed stable curve

$$(C, p_1, \ldots, p_g)$$

of genus 0 to the stable curve

$$\big(C \cup \bigcup_{i=1}^{g} E_i, p_0\big)$$

of genus g, where E_i are fixed non-isomorphic elliptic curves, attached to C at the points p_i, and $p_0 \in E_1$ is fixed as well.

FIGURE 1. $i(C, p_1, \ldots, p_g)$

Definition 4.2. Let $j \colon \widetilde{\mathcal{M}}_{2,1} \hookrightarrow \widetilde{\mathcal{M}}_{g,1}$ be the family of curves defined by sending a marked curve (C, p) to the marked stable curve

$$(C \cup C', p_0)$$

where (C', p', p_0) is a fixed Brill-Noether-general curve in $\widetilde{\mathcal{M}}_{g-2,2}$, attached nodally to (C, p) at p'.

FIGURE 2. $j(C, p)$

Definition 4.3. Fix Brill-Noether general curves

$$(C_1, p_1) \in \mathcal{M}_{h,1}$$
$$(C_2, p_2) \in \mathcal{M}_{g-h,1}$$

and let $C = C_1 \cup C_2$ be their nodal union along the p_i. Let $k_h \colon C_1 \hookrightarrow \widetilde{\mathcal{M}}_{g,1}$ be the map sending $p \in C_1$ to the marked curve (C, p).

4-B. **Computations on the Special Families.**

Lemma 4.4. *For the family*

$$i \colon \overline{\mathcal{M}}_{0,g} \hookrightarrow \widetilde{\mathcal{M}}_{g,1}$$

we have

$$\eta_* \alpha = \eta_* \beta = \eta_* \gamma = 0$$

Lemma 4.5. *For the family*

$$j \colon \widetilde{\mathcal{M}}_{2,1} \hookrightarrow \widetilde{\mathcal{M}}_{g,1}$$

we have

$$\eta_* \alpha = \frac{2dN(d - 2g + 2)}{3(g-1)}(3\psi - \lambda - \delta_1) + \frac{dN}{g-1}(\lambda + \delta_1 - 4\psi)$$

$$\eta_* \beta = \frac{dN}{g-1}(\lambda + \delta_1 - 4\psi)$$

$$\eta_* \gamma = \frac{-N\xi}{3(g-1)}(3\psi - \lambda - \delta_1),$$

where N and ξ are defined in the statement of Theorem 3.5.

Lemma 4.6. *For the family*

$$k_h \colon C_1 \hookrightarrow \widetilde{\mathcal{M}}_{g,1}$$

we have

$$\deg \eta_* \alpha = -d^2 N$$
$$\deg \eta_* \beta = -\big[2(g-h) - 1\big] dN$$
$$\deg \eta_* \gamma = -\big[rh + \tfrac{1}{2}r(r+1)\big] N$$

4-C. **Pull-Back Maps on Divisors.**

Lemma 4.7. *Let ϵ_i be the class of the closure of the locus on $\overline{\mathcal{M}}_{0,g}$ of stable curves with two components, the component containing the first marked point having i marked points.*

(a) *The classes ϵ_i are independent in $H^2(\overline{\mathcal{M}}_{0,g}; \mathbf{Q})$.*
(b) *For the family*

$$i \colon \overline{\mathcal{M}}_{0,g} \hookrightarrow \widetilde{\mathcal{M}}_{g,1}$$

we have the following pull-back map on divisor classes.

$$i^* \lambda = i^* \psi = i^* \delta_0 = 0$$
$$i^* \delta_i = \epsilon_i \qquad for\ i = 2, 3, \ldots, g-2$$
$$i^* \delta_1 = -\sum_{i=2}^{g-2} \frac{(g-i)(g-i-1)}{(g-1)(g-2)} \epsilon_i$$
$$i^* \delta_{g-1} = -\sum_{i=2}^{g-2} \frac{(g-i)(i-1)}{g-2} \epsilon_i$$

Lemma 4.8. *For the family*

$$j \colon \widetilde{\mathcal{M}}_{2,1} \hookrightarrow \widetilde{\mathcal{M}}_{g,1}$$

we have the following pull-back map on divisor classes.

$$j^* \lambda = \lambda \qquad\qquad\qquad j^* \psi = 0$$
$$j^* \delta_0 = \delta_0 \qquad\qquad\qquad j^* \delta_i = 0 \quad i = 1, 2, \ldots, g-3$$
$$j^* \delta_{g-2} = -\psi \qquad\qquad\qquad j^* \delta_{g-1} = \delta_1$$

Lemma 4.9. *For the family*

$$k_h \colon C_1 \hookrightarrow \widetilde{\mathcal{M}}_{g,1}$$

we have the following pull-back map on divisor classes.

$$\deg k_h^* \lambda = 0 \qquad\qquad\qquad \deg k_h^* \psi = 2h - 1$$
$$\deg k_h^* \delta_h = -1 \qquad\qquad\qquad \deg k_h^* \delta_{g-h} = 1$$
$$\deg k_h^* \delta_i = 0 \quad i \neq h, g-h$$

Proof of Theorem 3.5. Theorem 3.5 is now a consequence of the above lemmas. The main point is that the pull-backs of the classes $\eta_* \alpha$, $\eta_* \beta$, and $\eta_* \gamma$ to our special families coincide with the classes computed in Section 4-B. For example, to see this for $j^* \eta_* \gamma$, form the fiber the fiber square

$$
\begin{array}{ccc}
j^* \mathcal{G}_d^r & \xrightarrow{\ j'\ } & \mathcal{G}_d^r \\
{\scriptstyle \eta'} \downarrow & & \downarrow {\scriptstyle \eta} \\
\widetilde{\mathcal{M}}_{2,1} & \xrightarrow[\ j\ ]{} & \widetilde{\mathcal{M}}_{g,1}.
\end{array}
$$

Notice that although j is a regular embedding, j' need not be. Nonetheless, according to Fulton [7, Chapter 6], there is a refined Gysin homomorphism

$$
j^! \colon A_k(\mathcal{G}_d^r) \to A_{k-l}(j^* \mathcal{G}_d^r),
$$

where l is the codimension of j, which commutes with push-forward:

$$
\eta'_* j^! = j^* \eta_*.
$$

We need to check that

$$
j^! c_1(\mathcal{V}) \cap [\mathcal{G}_d^r] = c_1(j'^* \mathcal{V}) \cap [j^* \mathcal{G}_d^r].
$$

Since

$$
j^! c_1(\mathcal{V}) \cap [\mathcal{G}_d^r] = c_1(j'^* \mathcal{V}) \cap j^! [\mathcal{G}_d^r]
$$

[7, Proposition 6.3], it is enough to check that

$$
j^! [\mathcal{G}_d^r] = [j^* \mathcal{G}_d^r].
$$

Generalizing the dimension upper bound in [16, Corollary 5.9] to the multi-component case [14], we obtain

$$
\dim j^* \mathcal{G}_d^r = \dim \widetilde{\mathcal{M}}_{2,1}.
$$

This implies that the codimension of j' is equal to that of j, so the normal cone of j' is equal to the pull-back of the normal bundle of j, and the result follows.

Now, for example, to compute $\eta_* \gamma$, write

$$
\eta_* \gamma = a\lambda - \sum_{i=0}^{g-1} b_i \delta_i + c\psi
$$

Our goal is to solve for $a, b_0, b_1, \ldots, b_{g-1}, c$. Using Lemmas 4.6 and 4.9, we may solve for c and write b_{g-i} in terms of b_i. From Lemmas 4.4 and 4.7, we may further solve for $b_1, b_2, \ldots, b_{g-2}$ in terms of b_{g-1}. It remains to determine a, b_0, and b_{g-1}. This is done by pulling back to $\widetilde{\mathcal{M}}_{2,1}$, which has Picard number 3, and using Lemmas 4.5 and 4.8. The other push-forwards are computed similarly.   □

5. Proofs of Lemmas

In this section, we give proofs of the lemmas stated in Sections 4-B and 4-C.

Proof of Lemma 4.4. If $\overline{\mathcal{C}}_{0,g} \to \overline{\mathcal{M}}_{0,g}$ is the universal stable curve, then $i^*\widetilde{\mathcal{C}}_{g,1}$ is formed by attaching $\overline{\mathcal{M}}_{0,g} \times E_i$ to $\overline{\mathcal{C}}_{0,g}$ along the marked sections $\sigma_i \colon \overline{\mathcal{M}}_{0,g} \to \overline{\mathcal{C}}_{0,g}$. We have the following fiber square.

$$\begin{array}{ccc} i^*\mathcal{C}_d^r & \longrightarrow & i^*\widetilde{\mathcal{C}}_{g,1} \\ \downarrow & & \downarrow \\ i^*\mathcal{G}_d^r & \longrightarrow & \overline{\mathcal{M}}_{0,g} \end{array}$$

By the Plücker formula for $\mathbf{P}^1$, given $[C] \in \overline{\mathcal{M}}_{0,g}$, a limit linear series on $i(C)$ must have the aspect

$$(d - r - 1)p_i + |(r+1)p_i|$$

on each E_i. The line bundle $\mathcal{L} \to i^*\mathcal{C}_d^r$ is, therefore, the pull-back from $i^*\widetilde{\mathcal{C}}_{g,1}$ of the bundle which is given by

$$\pi_2^*\mathcal{O}_{E_1}(dp)$$

on $\overline{\mathcal{M}}_{0,g} \times E_1$ and is trivial on all other components. Thus $\alpha = \beta = 0$. The vector bundle $\mathcal{V} \subset \pi_*\mathcal{L}$ is trivial with fiber isomorphic to

$$H^0(\mathcal{O}_{E_1}((r+1)p)) \subset H^0(\mathcal{O}_{E_1}(dp))$$

so $\gamma = 0$ as well. $\qquad\qquad\square$

Before proving Lemma 4.5 we state an elementary result in Schubert calculus.

Lemma 5.1. [9, p. 266] *For integers r and d with $0 \le r \le d$, let*

$$X = \mathbf{G}(r, \mathbf{P}^d)$$

be the Grassmannian of r-planes in $\mathbf{P}^d$. For integers

$$0 \le b_0 \le b_1 \le \cdots \le b_r \le d - r,$$

let $\sigma_b = \sigma_{b_r,\ldots,b_0}$ be the corresponding Schubert cycle of codimension $\sum b_i$. Let $\zeta = \sigma_{1,1,\ldots,1,0}$ be the special Schubert cycle of codimension r. If k is an integer for which

$$rk + \sum_{i=0}^{r} b_i = \dim X = (r+1)(d-r),$$

then

$$\int_X \zeta^k \cdot \sigma_b = \frac{k!}{\prod_{i=0}^{r}(k-d+r+a_i)!} \prod_{0 \le i < j \le r} (a_j - a_i),$$

where $a_i = b_i + i$.

Proof of Lemma 4.5. Since $\overline{\mathcal{M}}_{2,1}$ is a smooth Deligne-Mumford stack, it is enough, by the moving lemma, to prove Lemma 4.5 for a family over a complete curve

$$B \hookrightarrow \widetilde{\mathcal{M}}_{2,1}$$

which intersects the boundary and Weierstrass divisors transversally. If $\pi \colon \mathcal{C} \to B$ is the universal stable genus-2 curve and $\sigma \colon B \to \mathcal{C}$ is the marked section, then $j^*\widetilde{\mathcal{C}}_{g,1}$ is formed by attaching $\mathcal{C}$ to $B \times C'$ along the marked section $\Sigma = \sigma(B) \subset \mathcal{C}$.

We begin by assuming that B is disjoint from the closure of the Weierstrass locus W. In this case we claim that

$$j^*\mathcal{G}_d^r \to B$$

is a trivial N-sheeted cover of the form $B \times X$, where is X a zero-dimensional scheme of length N. Indeed, for any curve (C, p) in $\widetilde{\mathcal{M}}_{2,1} \setminus W$ there are two (limit linear) $\mathfrak{g}_d^r$s on C with maximum ramification at p; the vanishing sequences are

$$a_1 = (d - r - 2, d - r - 1, \ldots, d - 4, d - 3, d),$$
$$a_2 = (d - r - 2, d - r - 1, \ldots, d - 4, d - 2, d - 1).$$

If C is smooth, the two linear series are

$$(d - r - 2)p + |(r + 2)p|$$

and

$$(d - r - 2)p + |rp + K_C|.$$

There are analogous series on nodal curves outside the closure of the Weierstrass locus. In the case of irreducible nodal curves, the sheaves are locally free.

For each of the two $\mathfrak{g}_d^r$s on C with maximum ramification at p, there are finitely many $\mathfrak{g}_d^r$s on C' with compatible ramification. Specifically, there are

$$\frac{(2g - 2 - d)N}{2(g - 1)}$$

of type a_1 and

$$\frac{dN}{2(g - 1)}$$

of type a_2, for total of N limit linear series counted with multiplicity. Since C' is fixed, the cover $j^*\mathcal{G}_d^r \to B$ is a trivial N-sheeted cover.

Consider a reduced sheet $B_1 \simeq B$ of type a_1. (We assume for simplicity that the sheet is reduced—the computation is the same in the general case.) Then the universal line bundle $\mathcal{L}$ on $j^*\mathcal{C}_d^r$ is given as

$$\mathcal{L} \cong \begin{cases} \mathcal{O}_{\mathcal{C}} & \text{on } \mathcal{C} \\ \pi_2^* L_1 & \text{on } B_1 \times C' \end{cases}$$

for some line bundle L_1 on C' of degree d. It follows that $\alpha = \beta = \gamma = 0$ on B_1.

Next consider a sheet $B_2 \simeq B$ of type a_2. Over $B_2 \times C'$ the universal line bundle $\mathcal{L}$ is isomorphic to $\pi_2^* L_2$ for some L_2 of degree d on C'. It remains to determine $\mathcal{L}$ over $\mathcal{C}$. Now $\omega_C(-2p)$ gives the correct line bundle for all $[C] \in B_2$; however, it has the wrong degrees on the components of the singular fibers. As our first approximation to $\mathcal{L}$ on $\mathcal{C}$ we take

$$\omega_{\mathcal{C}/B_2}(-2\Sigma)$$

Let $\Delta \subset \mathcal{C}$ be the pull-back of the divisor on $\mathcal{C}$ of curves of the form $C_1 \cup C_2$, where the C_i have genus one, and the marked points lie on different components. Then

$$\omega_{\mathcal{C}/B_2}(-2\Sigma + \Delta)$$

has the correct degree on the irreducible components on each fiber. It remains only to normalize our line bundle by pull-backs from the base B_2. In this case, $\mathcal{L}|_{\mathcal{C}}$ is required to be trivial along Σ since $\mathcal{L}$ is a pull-back from C' on the other component. If $\sigma\colon B_2 \to \mathcal{C}$ is the marked section, we let

$$\Psi = \sigma^* \omega_{\mathcal{C}/B_2}$$

be the tautological line bundle on B_2. Then

$$\sigma^* \mathcal{O}_\mathcal{C}(\Delta) \cong \mathcal{O}_{B_2}$$
$$\sigma^* \mathcal{O}_\mathcal{C}(\Sigma) \cong \Psi^\vee$$

It follows that on $\mathcal{C}$,

$$\mathcal{L} \cong \begin{cases} \omega_{\mathcal{C}/B_2}(-2\Sigma + \Delta) \otimes \pi^* \Psi^{\otimes -3} & \text{on } \mathcal{C} \\ \pi_2^* L_2 & \text{on } B_2 \times C'. \end{cases}$$

Thus, if we let

$$\omega = c_1(\omega_{\mathcal{C}/B_2})$$
$$\sigma = c_1(\mathcal{O}_\mathcal{C}(\Sigma))$$
$$\delta = c_1(\mathcal{O}_\mathcal{C}(\Delta))$$

on $\mathcal{C}$ and let

$$\psi = c_1(\Psi)$$

on B_2, then

$$c_1(\mathcal{L}) = \begin{cases} \omega - 2\sigma + \delta - 3\pi^*\psi & \text{on } \mathcal{C} \\ d\pi_2^* p & \text{on } B_2 \times C' \end{cases}$$

For the relative dualizing sheaf $\omega_{j^*\widetilde{\mathcal{C}}_{g,1}/B_2}$, we have

$$c_1(\omega_{j^*\mathcal{C}/B_2}) = \begin{cases} \omega + \sigma & \text{on } \mathcal{C} \\ (2(g-2)-1)\pi_2^* p & \text{on } B_2 \times C'. \end{cases}$$

To compute the products of these classes on $j^*\mathcal{C}_d^r$, recall the following formulas on $\pi \colon \overline{\mathcal{C}}_{2,1} \to \overline{\mathcal{M}}_{2,1}$

$$\pi_*\omega = 2 \qquad\qquad \pi_*\delta = 0 \qquad\qquad \pi_*\sigma = 1$$
$$\pi_*\omega^2 = 12\lambda - \delta_0 - \delta_1 \qquad \pi_*\sigma^2 = -\psi \qquad \pi_*\delta^2 = -\delta_1$$
$$\pi_*(\delta.\sigma) = 0 \qquad\qquad \pi_*(\omega.\delta) = \delta_1 \qquad \pi_*(\sigma.\omega) = \psi.$$

Then we compute

$$\alpha = \pi_*\big[c_1(\mathcal{L})^2\big] = 12\lambda - \delta_0 - 8\psi$$
$$\beta = \pi_*\big[c_1(\mathcal{L}) \cdot c_1(\omega)\big] = 12\lambda - \delta_0 - 8\psi$$

on B_2. Since the marked point lies on C', $\mathcal{V}$ is trivial on B_2, so $\gamma = 0$ on B_2.

Finally we consider the case where B (transversally) intersects the Weierstrass locus. In this case
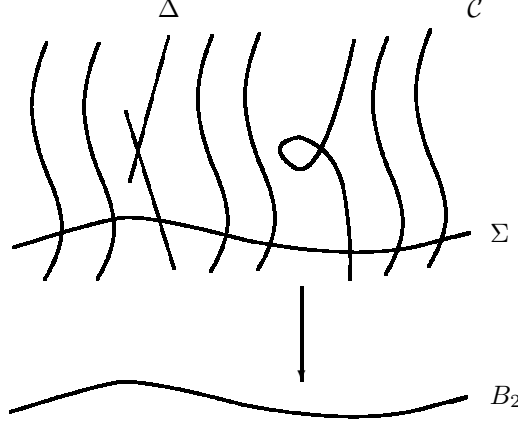
$$\eta \colon j^*\mathcal{G}_d^r \to B$$

is the union of a trivial N-sheeted cover of B and a 1-dimensional scheme lying over each point of the divisor W. It will suffice to compute α and γ on

$$G_d^r(j[C,p])$$

where C is a smooth genus-2 curve, and $p \in C$ is a Weierstrass point. (Note that β is automatically zero.)

There is a single $\mathfrak{g}_d^r$ on C with maximal ramification at p, namely

$$(d-r-2)p + |(r+2)p|,$$

FIGURE 3. The morphism $\mathcal{C} \to B_2$.

which has vanishing sequence

$$(d - r - 2, d - r - 1, \ldots, d - 4, d - 2, d).$$

We claim that we only need to consider components of $G_d^r(C \cup C')$ with this aspect on C. Indeed, any $\mathfrak{g}_d^r$ on C with ramification 1 less at p is still a subseries of $|dp|$. There will be finitely many corresponding aspects on C' so that, as before, $\alpha = \beta = \gamma = 0$ on these components.

It remains to consider the components of $G_d^r(C \cup C')$ where the aspect on C' has ramification $(0, 1, 2, 2, \ldots, 2)$ or more at p'. We are reduced to studying the one-dimensional scheme

$$S = G_d^r(C'; p', (0, 1, 2, 2, \ldots, 2)).$$

To simplify computations we specialize C' to a curve which is the union of $\mathbf{P}^1$ with $g - 2$ elliptic curves $E_1, \ldots, E_{g-2}$ attached at general points $p_1, \ldots, p_{g-2}$, and where the marked point p_0 lies on E_1, and the point of attachment p' lies on the $\mathbf{P}^1$. There will be two types of components of S: those on which the aspects on the
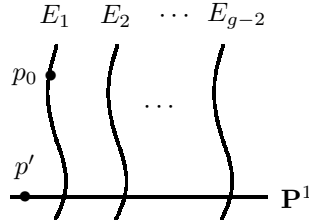
FIGURE 4. The curve C'.

E_i are maximally ramified at p_i, and those on which the aspect on one E_i varies. Again, as in the proof of Lemma 4.4, we need only consider the latter case.

Assume that for some i, the ramification at p_i of the $\mathfrak{g}_d^r$ on E_i is one less than maximal. There are two possibilities: either the series is of the form

$$(d - r - 1)p_i + |rp_i + q| \qquad \text{for } q \in E_i,$$

which for $q \neq p_i$ imposes on the $\mathbf{P}^1$ the ramification condition

$$(1, 1, \ldots, 1),$$

or the $\mathfrak{g}_d^r$ is a subseries of

$$(d - r - 2)p_i + |(r + 2)p_i|$$

containing

$$(d - r)p_i + |rp_i|,$$

which generically imposes on the $\mathbf{P}^1$ the ramification condition

$$(0, 1, 1, \ldots, 1, 2).$$

In the first case the components are parameterized by E_i, and we compute that $\alpha = -2$ on each such irreducible component, irrespective of whether $i = 1$ or not. By Grothendieck-Riemann-Roch, $\gamma = -1$ when $i = 1$ and is zero otherwise. In the second case the $\mathfrak{g}_d^r$s are parameterized by a $\mathbf{P}^1$. Because the line bundle is constant, $\alpha = 0$. On each such $\mathbf{P}^1$, the vector bundle $\mathcal{V}$ may be viewed as the tautological bundle of rank $r+1$ on the Grassmannian of vector subspaces of a fixed vector space of dimension $r + 2$ containing a subspace of dimension r. It follows that $\gamma = -1$ on each $\mathbf{P}^1$.

Let $X = \mathbf{G}(r, \mathbf{P}^d)$ be the Grassmannian of r-planes in $\mathbf{P}^d$. Let

$$\zeta = \sigma_{1,1,\ldots,1,0}$$

be the special Schubert cycle of codimension r. Collecting our calculations, we have that on $G_d^r(C \cup C')$,

$$\alpha = -2(g - 2) \int_X \sigma_{2,2,\ldots,2,1,0} \cdot \sigma_{1,1,\ldots,1} \cdot \zeta^{g-3}$$

$$= -2(g - 2) \int_X \sigma_{3,3,\ldots,3,2,1} \cdot \zeta^{g-3},$$

and

$$\gamma = - \int_X \sigma_{2,2,\ldots,2,1,0} \cdot (\sigma_{1,1,\ldots,1} + \sigma_{2,1,1,\ldots,1,0}) \cdot \zeta^{g-3}$$

$$= - \int_X \sigma_{2,2,\ldots,2,1,0} \cdot (\sigma_{1,0,0,\ldots,0} \cdot \zeta) \cdot \zeta^{g-3}$$

$$= - \int_X (\sigma_{3,2,2,\ldots,2,1,0} + \sigma_{2,2,\ldots,2,0} + \sigma_{2,2,\ldots,2,1,1}) \cdot \zeta^{g-2}$$

$$= - \int_X (\sigma_{3,2,2,\ldots,2,1,0} + \zeta^2) \cdot \zeta^{g-2}$$

$$= - \int_X \sigma_{3,2,2,\ldots,2,1,0} \cdot \zeta^{g-2} - \int_X \zeta^g.$$

From Lemma 5.1 we compute,

$$\alpha = \frac{-2d(2g - 2 - d)N}{3(g - 1)}$$

$$\gamma = \frac{-\xi N}{3(g - 1)}.$$

Since the class of the Weierstrass locus in $\widetilde{\mathcal{M}}_{2,1}$ is $3\psi - \lambda - \delta_1$, the lemma follows. $\square$

Proof of Lemma 4.6. Because the curves (C_i, p_i) are Brill-Noether general, $k_h^* \mathcal{G}_d^r$ is a trivial N-sheeted cover of C_1 of the form $C_1 \times X$, where X is a zero-dimensional scheme of length N. Fix a sheet $G \cong C_1$ in $k_h^* \mathcal{G}_d^r$; this choice corresponds to aspects

$$V_i \subset H^0(C_i, L_i)$$

where L_i are degree-d line bundles on C_i. If $(a_0, a_1, \ldots, a_r)$ is the vanishing sequence of V_1 at p_1, then we know that

$$0 = \rho(h, r, d) - \sum_{i=0}^{r}(a_i - i)$$

$$= (r+1)(d-r) - hr - \sum_{i=0}^{r} a_i + \frac{1}{2}r(r+1),$$

so

(1) $$\sum_{i=0}^{r} a_i = (r+1)d - \frac{1}{2}r(r+1) - hr.$$

Let $\mathcal{C}_1$ be the blow-up of $C_1 \times C_1$ at (p_1, p_1), E the exceptional divisor, and e its first Chern class. We may construct the universal curve $k_h^* \widetilde{\mathcal{C}}_{g,1} \to C_1$ by attaching $C_1 \times C_2$ to $\mathcal{C}_1$ along $C_1 \times \{p_2\}$ and the proper transform of $C_1 \times \{p_1\}$. Over the sheet G, the universal line bundle $\mathcal{L}$ on $k_h^* \mathcal{C}_d^r$ is

$$\pi_2^* L_1 \otimes \mathcal{O}_{\mathcal{C}_1}(-dE) \otimes \pi_1^* L_1^\vee(dp_1)$$

on $\mathcal{C}_1$ and

$$\pi_2^* L_2(-dp_2) \otimes \pi_1^* L_1^\vee$$

on $C_1 \times C_2$. Thus

$$c_1(\mathcal{L}) = \begin{cases} d\pi_2^* p - de & \text{on } \mathcal{C}_1 \\ -d\pi_1^* p & \text{on } C_1 \times C_2. \end{cases}$$

The relative dualizing sheaf $\omega_{k_h^* \widetilde{\mathcal{C}}_{g,1}/C_1}$ is isomorphic to

$$\pi_2^* \omega_{C_1} \otimes \mathcal{O}_{\mathcal{C}}(E) \otimes \pi_1^* \mathcal{O}_{C_1}(-p_1)$$

on $\mathcal{C}_1$ and

$$\pi_2^* \omega_{C_2}(p_2)$$

on $C_1 \times C_2$. We have

$$c_1(\omega) = \begin{cases} -\pi_1^* p + (2h-2)\pi_2^* p + e & \text{on } \mathcal{C}_1 \\ (2(g-h)-1)\pi_2^* p & \text{on } C_1 \times C_2. \end{cases}$$

Thus, on G,

$$\deg \alpha = c_1(\mathcal{L})^2 = -d^2$$

$$\deg \beta = c_1(\mathcal{L}) \cdot c_1(\omega) = -d\big[2(g-j)-1\big].$$

The formulas for $\eta_* \alpha$ and $\eta_* \beta$ now follow.

To calculate γ on G, notice that it suffices to compute $c_1(\mathcal{V}')$, where

$$\mathcal{V}' = \mathcal{V} \otimes L_1(-dp_1)$$

is a sub-bundle of

$$\pi_{1*}(\pi_2^* L_1(-dE)).$$

We claim there is a bundle isomorphism

$$\bigoplus_{j=0}^{r} \mathcal{O}_{C_1}((a_j - d)p_1) \xrightarrow{\cong} \mathcal{V}'$$

To describe the map, pick a basis $(\sigma_0, \sigma_1, \ldots, \sigma_r)$ of $V_1 \subset H^0(L_1)$ with σ_i vanishing to order a_i at p_1. Given local sections τ_i of $\mathcal{O}_{C_1}((a_i - d)p_1)$, let the image of $(\tau_0, \tau_1, \ldots, \tau_r)$ be the section

$$\sum_{i=0}^{r} \sigma_i \tau_i$$

of $\mathcal{V}'$. This is clearly an isomorphism away from p_1 and is checked to be an isomorphism over p_1 as well. Using (1), we have that on G,

$$\deg \gamma = \deg \mathcal{V}' = \sum_{i=0}^{r}(a_i - d)$$
$$= -\frac{1}{2}r(r+1) - rh$$

which finishes the proof of the lemma. □

Proof of Lemma 4.7. To prove the independence of the ϵ_i, consider the curves

$$B_j \hookrightarrow \overline{\mathcal{M}}_{0,g}$$

for $j = 2, 3, \ldots, g-3$ given by taking a fixed stable curve in ϵ_j and moving a marked point on the component with $g - j$ marked points. Let $B_1 \hookrightarrow \overline{\mathcal{M}}_{0,g}$ be the curve given by moving the first marked point along a fixed smooth curve. The intersection matrix

$$(\epsilon_i \cdot B_j)$$

is

$$\begin{pmatrix} g-1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & \cdots & 0 & 0 & g-3 \\ 0 & -1 & 1 & 0 & \cdots & 0 & 0 & g-4 \\ \vdots & & & & \ddots & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 & 3 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}$$

where the rows correspond to the B_j for $j = 1, 2, \ldots, g-3$, and the columns to the ϵ_i for $i = 2, 3, \ldots, g-2$. Since this matrix is non-singular, the first part of the lemma follows.

To derive the formula for the pull-back, we follow Harris and Morrison [12, Section 6.F]. Let B be a smooth projective curve, $\pi \colon \mathcal{C} \to B$ a 1-parameter family of curves in $\overline{\mathcal{M}}_{0,g}$ transverse to the boundary strata. Then π has smooth total space, and the fibers of π have at most two irreducible components. Let

$$\sigma_i \colon B \to \mathcal{C}$$

be the marked sections. Denote by Σ_i the image curve $\sigma_i(B)$ in $\mathcal{C}$. Then on B,

$$\delta_1 = \Sigma_1^2$$
$$\delta_{g-1} = \sum_{j=2}^{g} \Sigma_j^2,$$

where we are using D^2 to denote $\pi_*(D^2)$ for a divisor D on $\mathcal{C}$. We now contract the component of each reducible fiber which meets the section Σ_1. If $\overline{\Sigma}_j$ is the image of Σ_j under this contraction, then we have

$$\Sigma_1^2 = \overline{\Sigma}_1^2 - \sum_{i=2}^{g-2} \epsilon_i$$

$$\sum_{j=2}^{g} \Sigma_j^2 = \sum_{j=2}^{g} \overline{\Sigma}_j^2 - \sum_{i=2}^{g-2}(i-1)\epsilon_i$$

The $\overline{\Sigma}_j$ are sections of a $\mathbf{P}^1$-bundle, so

$$0 = (\overline{\Sigma}_j - \overline{\Sigma}_k)^2 = \overline{\Sigma}_j^2 + \overline{\Sigma}_k^2 - 2\overline{\Sigma}_j \cdot \overline{\Sigma}_k$$

Thus

$$(g-2)\sum_{j=2}^{g} \overline{\Sigma}_j^2 = \sum_{2 \le j,k \le g} (\overline{\Sigma}_j^2 + \overline{\Sigma}_k^2) = 2 \sum_{2 \le j,k \le g} \overline{\Sigma}_j \cdot \overline{\Sigma}_k$$

$$= 2\sum_{i=2}^{g-2} \binom{i-1}{2}\epsilon_i.$$

It follows that

$$\delta_{g-1} = \sum_{i=2}^{g-2} \left[\frac{(i-1)(i-2)}{g-2} - (i-1) \right] \epsilon_i = \sum_{i=2}^{g-2} \frac{(i-1)(i-g)}{g-2}\epsilon_i$$

Similarly, we can show that

$$\delta_1 + \delta_{g-1} = \sum_{i=2}^{g-2} \frac{i(i-g)}{g-1}\epsilon_i$$

so the formula for δ_1 follows as well. $\qquad\square$

The proofs of Lemmas 4.8 and 4.9 are straightforward, so we omit them.

6. APPENDIX

Proposition 6.1. *Let Σ be a set of 21 general points in $\mathbf{P}^2$ and let $S = \mathrm{Bl}_\Sigma \mathbf{P}^2$ be the blow-up of $\mathbf{P}^2$ at Σ. If H is the line class on S and $E_1, \ldots, E_{21}$ are the exceptional divisors, then the linear system*

$$\left| 13H - 2\sum_{j=1}^{9} E_j - 3\sum_{k=10}^{21} E_k \right|$$

on S contains a smooth connected curve.

Proof. We begin by showing that it is enough to exhibit a single set of 21 points over a finite field for which the above statement is true.

Let $\mathrm{Hilb}_{\mathbf{Z}}^k \mathbf{P}^2$ be the Hilbert scheme of k points in $\mathbf{P}^2$, and let

$$\Sigma_k \subset \mathrm{Hilb}_{\mathbf{Z}}^k \mathbf{P}^2 \times \mathbf{P}^2$$

be the universal subscheme. Let

$$B \subset \mathrm{Hilb}_{\mathbf{Z}}^9 \mathbf{P}^2 \times \mathrm{Hilb}_{\mathbf{Z}}^{12} \mathbf{P}^2$$

be the irreducible open subset over which the composition

$$\Sigma = \pi_1^{-1}\Sigma_9 \cup \pi_2^{-1}\Sigma_{12} \longrightarrow \mathrm{Hilb}_{\mathbf{Z}}^9 \mathbf{P}^2 \times \mathrm{Hilb}_{\mathbf{Z}}^{12} \mathbf{P}^2 \times \mathbf{P}^2$$

$$\downarrow$$

$$\mathrm{Hilb}_{\mathbf{Z}}^9 \mathbf{P}^2 \times \mathrm{Hilb}_{\mathbf{Z}}^{12} \mathbf{P}^2$$

is étale, where π_1 and π_2 are the obvious projections. Let

$$\pi \colon S = \mathrm{Bl}_\Sigma \mathbf{P}_B^2 \to B$$

be the smooth surface over B whose fibers are blow-ups of $\mathbf{P}^2$ at 21 distinct points. If E_9 and E_{12} are the exceptional divisors, let

$$\mathcal{L} = \mathcal{O}_S(13H - 2E_9 - 3E_{12}).$$

We may further restrict B to an open over which $\pi_*\mathcal{L}$ is locally free of rank at least 6.

If

$$\mathcal{C} \subset \mathbf{P}\pi_*\mathcal{L} \times_B S$$

is the universal section, then the projection

$$\mathcal{C} \to \mathbf{P}\pi_*\mathcal{L}$$

is flat, so it suffices to find a single smooth fiber in order to conclude that the general fiber is smooth. To this end we use Macaulay 2 [8] and work over a finite field.

```
i1 : S = ZZ/137[x,y,z];
```

Following Shreyer and Tonoli [17], we realize our points in $\mathbf{P}^2$ as a determinental subscheme.

```
i2 : randomPlanePoints = (delta,R) -> (
        k:=ceiling((-3+sqrt(9.0+8*delta))/2);
        eps:=delta-binomial(k+1,2);
        if k-2*eps>=0
        then minors(k-eps,
          random(R^(k+1-eps),R^{k-2*eps:-1,eps:-2}))
        else minors(eps,
          random(R^{k+1-eps:0,2*eps-k:-1},R^{eps:-2})));
```

```
i3 : distinctPoints = (J) -> (
        singJ = minors(2, jacobian J) + J;
        codim singJ == 3);
```

Let Σ_9 and Σ_{12} be our subsets of 9 and 12 points, respectively.

```
i4 : Sigma9 = randomPlanePoints(9,S);
```

```
o4 : Ideal of S
```

```
i5 : Sigma12 = randomPlanePoints(12,S);

o5 : Ideal of S

i6 : (distinctPoints Sigma9, distinctPoints Sigma12)

o6 = (true, true)

o6 : Sequence
```

Their union is Σ.

```
i7 : Sigma = intersect(Sigma9, Sigma12);

o7 : Ideal of S

i8 : degree Sigma

o8 = 21
```

Next we construct the 0-dimensional subscheme Γ whose ideal consists of curves double through points of Σ_9 and triple through points of Σ_{12}.

```
i9 : Gamma = saturate intersect(Sigma9^2, Sigma12^3);

o9 : Ideal of S
```

Let us check that Γ imposes the expected number of conditions $(9 \cdot 3 + 12 \cdot 6 = 99)$ on curves of degree 13.

```
i10 : hilbertFunction (13, Gamma)

o10 = 99
```

Pick a random curve C of degree 13 in the ideal of Γ.

```
i11 : C = ideal (gens Gamma
                    * random(source gens Gamma, S^{-13}));

o11 : Ideal of S
```

We check that C is irreducible.

```
i12 : # decompose C
```

```
o12 = 1
```

To check smoothness, let C_{sing} be the singular locus of C.

```
i13 : Csing = (ideal jacobian C) + C;
```

```
o13 : Ideal of S
```

```
i14 : codim Csing
```

```
o14 = 2
```

A double point will contribute 1 to the degree of C_{sing} if it is transverse and more otherwise. Similarly, a triple point will contribute 4 to the degree of C_{sing} if it is transverse and more otherwise. So for C to be smooth in the blow-up, we must have that
$$\deg C_{\mathrm{sing}} = 9 + 4 \cdot 12 = 57$$

```
i15 : degree Csing
```

```
o15 = 57
```

$\square$

References

[1] Allen B. Altman and Steven L. Kleiman, *Compactifying the Picard scheme*, Adv. in Math. **35** (1980), no. 1, 50–112.

[2] Pierre Deligne and David Mumford, *The irreducibility of the space of curves of given genus*, Inst. Hautes Études Sci. Publ. Math. (1969), no. 36, 75–109.

[3] David Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995, With a view toward algebraic geometry.

[4] David Eisenbud and Joseph Harris, *Limit linear series: basic theory*, Invent. Math. **85** (1986), no. 2, 337–371.

[5] Gavril Farkas, *Syzygies of curves and the effective cone of $\overline{\mathcal{M}}_g$*, to appear in Duke Mathematical Journal, arXiv:math.AG/0503498.

[6] Gavril Farkas and Mihnea Popa, *Effective divisors on $\overline{\mathcal{M}}_g$, curves on K3 surfaces and the Slope Conjecture*, J. Algebraic Geom. **14** (2005), no. 2, 241–267, arXiv:math.AG/0305112.

[7] William Fulton, *Intersection theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 2, Springer-Verlag, Berlin, 1984.

[8] Daniel R. Grayson and Michael E. Stillman, *Macaulay 2, a software system for research in algebraic geometry*, Available at http://www.math.uiuc.edu/Macaulay2/.

[9] Phillip Griffiths and Joseph Harris, *On the variety of special linear systems on a general algebraic curve*, Duke Math. J. **47** (1980), no. 1, 233–272.

[10] John Harer, *The second homology group of the mapping class group of an orientable surface*, Invent. Math. **72** (1983), no. 2, 221–239.

[11] Joseph Harris, *On the Severi problem*, Invent. Math. **84** (1986), no. 3, 445–461.

[12] Joseph Harris and Ian Morrison, *Slopes of effective divisors on the moduli space of stable curves*, Invent. Math. **99** (1990), no. 2, 321–355.

[13] Joseph Harris and David Mumford, *On the Kodaira dimension of the moduli space of curves*, Invent. Math. **67** (1982), no. 1, 23–88, with an appendix by William Fulton.

[14] Deepak Khosla and Brian Osserman, *A limit linear series moduli stack*, in preparation.

[15] Brian Osserman, *A limit linear series moduli scheme*, to appear in Annales de l'Institut Fourier, arXiv:math.AG/0407496.

[16] ———, *Linked grassmannians and crude limit linear series*, unpublished.

[17] Frank-Olaf Schreyer and Fabio Tonoli, *Needles in a haystack: special varieties via small fields*, Computations in algebraic geometry with Macaulay 2, Algorithms Comput. Math., vol. 8, Springer, Berlin, 2002, pp. 251–279.

Department of Mathematics, University of Texas at Austin, 1 University Station C1200, Austin, Texas 78712, USA

E-mail address: dkhosla@math.utexas.edu